The proof of the sufficiency is similar to that for the case $n = 2$.

To these two theorems may be added a third related theorem.

THEOREM III. *The necessary and sufficient condition that*

$$\int_{(C)} Pdx + Qdy \tag{8}$$

shall vanish, where C is any closed contour in a region in which P and Q are continuous, is that $Pdx + Qdy$ shall be an exact differential.

For if the line integral is zero and C is arbitrary, then

$$\int_{x_0, y_0}^{x, y} Pdx + Qdy$$

is a function of x and y only, and does not depend on the path.

That is,

$$\int_{x_0, y_0}^{x, y} Pdx + Qdy = v(x, y).$$

Therefore

$$P = \frac{\partial v}{\partial x}, \qquad Q = \frac{\partial v}{\partial y}$$

and the diffential is exact.

And again if $Pdx + Qdy$ is exact, it must be of the form du where u is the common value of the two expressions in (2). But the u as there defined is continuous in x and y, and therefore the total algebraic variation about a closed contour is zero.

The form of the integral expressions appearing in (2) and (6) is usually given as the formula for the integral when the differential is exact, but so far as I have been able to find the invariance of these expressions under cyclic interchange of notation has not been given as a criterion for exactness.

II. THE TEACHING OF LIMITS IN THE HIGH SCHOOL.[1]

By J. V. McKELVEY, Iowa State College.

The title of the present paper is to some extent either misleading or noncommittal. To make our purpose somewhat clearer, it may be stated that we hold no brief either for or against the teaching of limits in preparatory schools. We intend, rather, to state the results of several years' observation of high-school students during their early years in college particularly in regard to their understanding of limiting operations in the most elementary sense of the word. We open this discussion with whatever apologies may be necessary for saying some things that, perhaps, everybody knows.

To plunge rather abruptly into the midst of the question, we note that

[1] Read before the Iowa Academy of Science, April 24, 1920.

practically every high school graduate, except those who have avoided mathematics entirely, has heard of a limit and knows or thinks he knows what happens when a variable approaches a limit. If asked what is meant by a variable approaching a limit, he replies with the utmost assurance that if a variable approaches continually nearer and nearer to some constant to which it can never become equal, that constant is the limit of the variable. This is a most beautiful conception indeed. It would be a profoundly admirable one except for the fact that it is as totally unmathematical as anything could possibly be,—in the sense that the first part of the statement is insufficient and the second part is unnecessary. Nevertheless, the above reply may be taken as a fair composite statement of a college freshman's idea of a limit. This being the case, the question logically arises, where did he get it? The thousands of young people entering college each year are too nearly of one mind as to the definition of a limit for us to believe that their opinions are merely the result of accidental or spontaneous development in immature minds. Educational accidents do not happen with such regularity and persistence. If we discard the hypothesis that it just happened so, we are forced to the conclusion that somebody taught it to them.

In discussing the teaching proposition one is led to the consideration of both personnel and text-books. As regards personnel, we must with undisguised embarrassment admit that there are teachers of mathematics in high schools, normal schools, colleges and universities who either apologize for or openly accept the idea of a limit that we have just recognized as the almost unanimous choice of college freshmen. It is neither a student's place nor a scholar's to say that a definition is wrong but it is his privilege to believe that certain conceptions of fundamental ideas, used as definitions, are both useless and clumsy. The notion that a variable must of necessity regard its limit as a *sanctum sanctorum* into the privacy of which it dare not intrude is one that has neither defense nor excuse in mathematical argument. The writer has yet to learn of a single problem involving the theory of limits in which it is of the slightest consequence whether the variable reaches its limit or not. It is very difficult to understand why any definition of a limit that excludes the most favorable case in the limiting argument should find such wide acceptance among intelligent people. There seems to be abundant evidence in support of the statement that many persons attempt to teach mathematics who have no conception whatever of an infinitesimal except that it is some strange, unnameable sort of quantity but desperately small. This very unfortunate state of affairs is probably in large measure due to the fact that instructors exist who teach, letter by letter, the subject matter of their text-books with a reverence born of fear and uncertainty.

In discussing text-books, one must admit that many of them are in some respects a great handicap to the novice who takes them too seriously. Not long ago, the writer took occasion to examine a single shelf of about thirty elementary text-books in mathematics in regard to their definitions of a limit. A considerable number of them were designed for first and second year work and of course did not mention limits at all. Five of the remainder defined the limit as a

constant to which the variable could never become equal but such that the difference between the variable and the constant could become ever so small. A sixth, we blush to relate, merely defined zero as "nothing" or something less than "epsilon." Six text-books out of thirty contained these useless and confusing notions. The percentage should be stated somewhat higher than six out of thirty, for some of the books made no mention of limits whatever. The title pages indicate that the authors came from state normal schools, technical institutes, agricultural colleges and universities.

A plausible explanation of the persistence of these unfortunate conceptions is found in the fact that most students are introduced to the idea of a limit in terms of geometry. The time worn straight line illustration in which the point P moves from A toward B taking the positions P_1, P_2, P_3, etc., where P_i bisects the segment $P_{i-1}B$, is most popular. This may be because the illustration is easy, or perhaps because it is graphic. The one bit of information that the student gets out of this illustration and which eventually excludes every other feature of the argument is that the point P can never arrive at the point B. In this he is of course absolutely right, but so far as the limiting operation is concerned his information is just about as valuable as the discovery by a football coach that his star halfback had gray eyes instead of brown ones. In the above illustration of a limit the student should be taught that if C is a point on AB such that CB is arbitrarily small, then under the given law of motion P_i can be placed between C and B, *i.e.*, P_iB can be made less than CB. Sometimes the student's preparation may not be sufficient to make a rigorous proof of this fact either possible or desirable. In such cases, a few numerical examples will illustrate the principle so that the proper sequence of the operations may be understood. A satisfactory proof may be given when the student becomes acquainted with logarithms.

The various proportionality theorems, proofs concerning the areas and arcs of circles, together with a number of volume and surface problems which are discussed in our elementary plane and solid geometries constitute the subject matter from which the majority of students derive their notion of a limit. This is a particularly unfortunate circumstance because these are all cases in which the variables do not reach their limits. It is not surprising that the student thinks this ever present fact is of some consequence in the argument.

The idea that a point may take various positions on a straight line under such a law of motion that it can never reach the end of the line, or that the area of a polygon inscribed in a circle may be continually increased without being made larger than a certain fixed quantity, or in general the idea that any operation may continue indefinitely without reaching a specified goal so impresses or oppresses the beginner in mathematics that his reasoning powers seemingly cease to function so far as the essential argument in the case is concerned. Hence it is the writer's opinion that the above most vicious feature of the study of limits in geometry should be avoided by every means within the law. Can it be done? It can be done if, and only if, we give the student something else to think about.

The writer believes that limits should be taught entirely from the standpoint of inequalities. If a variable x assumes a sequence of values such that $|x - a|$ becomes and remains less than a pre-assigned positive number which is arbitrarily small, x is said to approach a as a limit. This is a quite generally accepted form of the definition. Its application depends absolutely and finally on the existence or non-existence of a certain inequality. The fact should be definitely emphasized that the positive number is assigned *first*. If after that the variable takes such values that the prescribed inequality exists, the variable has a limit, otherwise not.

This principle of the "order of choice" can not be over emphasized. It must be first the epsilon, then the variable. If this sequence is disturbed, the limiting argument breaks down completely. A variety of illustrations may be necessary to drive the principle home and make it stick. Numerical examples can be used to advantage. Instructive exercises may be given in finding the largest permissible numerical error in determining a required number so that the percentage error should be less than a specified value. Such examples will illustrate merely the skeleton of the argument. A constant effort should be made to induce beginners to waive temporarily any scriptural convictions they may have that the first should be last and the last should be first and to learn, in their mathematical reasoning at least, to put first things first and last things last.

III. Geometric Proof of the Law of Tangents.

By C. A. Epperson, Northeast Missouri State Teachers College.

Let $a > b$. Draw CD the bisector of the external angle at C (to meet BA produced at D) and CF the bisector of the angle C (meeting AB in F). Then CF is perpendicular to CD. Draw AN $(= w)$ and BM $(= y)$ parallel to FC, meeting DC in N and M respectively. $\angle BCM = \angle ACN = (A + B)/2$, and $\angle ADC = (A - B)/2$. Then

$$\frac{a}{b} = \frac{BD}{AD} = \frac{MD}{ND}.$$

By composition and division

$$\frac{a+b}{a-b} = \frac{MD + ND}{MD - ND} = \frac{MD + ND}{MC + CN} = \frac{(y + w)\cot\dfrac{A-B}{2}}{(y+w)\cot\dfrac{A+B}{2}};$$

$$\therefore \quad \frac{a+b}{a-b} = \frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}.$$